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**LETTER TO THE EDITOR**

# Space–times with a group of motions on null hypersurfaces

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**Abstract.** The energy conditions imply that null-homogeneous solutions of the Einstein field equations admit a non-expanding shear-free normal null congruence, and that these space–times are algebraically special.

By definition, spatially homogeneous space–times admit a group of motions $G_3$ acting on three-dimensional space-like surfaces of transitivity. Here we consider space–times in which a group of motions acts transitively on *null* hypersurfaces $N_3$; we call these space–times null-homogeneous. Petrov's book (Petrov 1966) contains a complete list of the normal-hyperbolic Riemannian spaces $V_4$ admitting a group $G_3$ on null hypersurfaces $N_3$. For all Bianchi types compatible with the Lorentzian signature, the special line elements and the corresponding Killing vectors have been given. Moreover, Petrov determined the subclass of these metrics which even admits higher-dimensional groups, $G_4$ and $G_5$, on $N_3$. In the papers on null-homogeneous space–times, the group is usually supposed to be a multiply transitive group (Lauten and Ray 1975, 1977, Ray 1976, MacCallum 1979, Crade and Hall 1979). The following considerations are not restricted to that case. We show that the energy conditions impose severe restrictions on null-homogeneous space–times and exclude some of the metrics given by Petrov. The plane waves are the only null-homogeneous solutions of the Einstein–Maxwell equations.

We suppose that the surfaces of transitivity (group orbits) of $G_3$ are null hypersurfaces $N_3$. At any point of $N_3$ there exists a single null direction. The associated normal null congruence

$$k_a = u_{,a}, \qquad k_a k^a = 0, \tag{1}$$

is automatically geodesic. The hypersurfaces $N_3$ can be parametrised by $u$ ($u$ being constant in each $N_3$). Three independent Killing vectors $\xi^a_A$ ($A = 1 \ldots 3$) lie in $N_3$ so that they are orthogonal to $k^a$,

$$\xi^a_A k_a = 0, \qquad \xi_{A(a;b)} = 0, \qquad k^a = C^A(x)\xi^a_A. \tag{2}$$

The Lie derivatives of $k^a$ with respect to $\xi^a_A$ vanish,

$$\mathcal{L}_A k^a \equiv \xi^b_A k^a_{;b} - k^b \xi^a_{A;b} = 0, \tag{3}$$

and quantities obtained by metric operations on $k^a$ have zero Lie derivatives; in particular

$$k^b_{;b;a}\xi^a_A = 0 = k^b_{;b;a}k^a. \tag{4}$$

We introduce a complex null tetrad $(k^a, l^a, t^a, \bar{t}^a)$, all scalar products being zero except

$$-k_a l^a = 1 = t_a \bar{t}^a. \tag{5}$$

The complex conjugate vectors $t^a$ and $\bar{t}^a$ are tangent to $N_3$ whereas the real null vector $l^a$ points away.

A physically reasonable energy–momentum tensor $T_{ab}$ has to obey the following conditions (see Hawking and Ellis 1973).

(i) The local energy density measured by any observer with four-velocity $u^a$ is non-negative,

$$T_{ab} u^a u^b \geqslant 0 \tag{6}$$

(weak energy condition).

(ii) In addition to (6), the local energy flow vector $q^a$ with respect to any observer is non-spacelike,

$$q_a q^a \leqslant 0, \qquad q^a \equiv T^{ab} u_b \tag{7}$$

(dominant energy condition).
By continuity, these conditions must still be true if we replace $u^a$ by the null vector $k^a$. Hence we impose the conditions

$$T_{ab} k^a k^b \geqslant 0, \tag{8}$$

$$T_{ab} T^a{}_c k^b k^c \leqslant 0. \tag{9}$$

From (4) and the Ricci identity we obtain

$$R_{ab} k^a k^b = k^b{}_{;a;b} k^a - k^b{}_{;b;a} k^a = -k_{a;b} k^{a;b}. \tag{10}$$

In terms of the shear and expansion of $k^a$, denoted by $\sigma$ and $\rho$ respectively, equation (10) reads

$$R_{ab} k^a k^b = -2(\sigma\bar{\sigma} + \rho^2) \leqslant 0, \tag{11}$$

$$\sigma \equiv k_{a;b} t^a t^b, \qquad \rho \equiv k_{a;b} t^a \bar{t}^b = \bar{\rho}. \tag{12}$$

Comparing the inequalities (8) and (11), we conclude that the relations

$$R_{ab} k^a k^b = 0 \tag{13}$$

$$\sigma = 0 = \rho \tag{14}$$

hold. The normal null congruence $k^a$ must be non-expanding and shear-free. Therefore the covariant derivative $k_{a;b}$ can be expressed as

$$k_{a;b} = k_a p_b + k_b p_a, \qquad p_a k^a = 0, \tag{15}$$

where $p^a$ is a vector in $N_3$.

With (13) and (14), the condition (9) takes the form

$$R_{ab} R^a{}_c k^b k^c = k^b{}_{;a;b} k^{c;a}{}_{;c} \leqslant 0. \tag{16}$$

A short calculation, however, leads to

$$\begin{aligned} &k^b{}_{;a;b} k^{c;a}{}_{;c} = \dot{p}_a \dot{p}^a = 2|R_{ab} t^a k^b|^2 \geqslant 0, \\ &\dot{p}_a \equiv p_{a;b} k^b, \qquad \dot{p}_a k^a = 0. \end{aligned} \tag{17}$$

The comparison of the inequalities (16) and (17) gives us the additional information

$$\dot{p}_a\dot{p}^a = 0 = R_{ab}t^a k^b, \tag{18}$$

i.e. $\dot{p}^a$ is parallel to $k^a$,

$$\dot{p}_{[a}k_{b]} = 0, \tag{19}$$

and $k^a$ is a Ricci eigendirection,

$$R_{ab}k^a k^b = 0 = R_{ab}t^a k^b \Leftrightarrow k_{[c}R_{a]b}k^b = 0. \tag{20}$$

Therefore, according to the Goldberg–Sachs (1962) theorem, the geodesic shear-free null congruence $k^a$ is a repeated principal null direction of the Weyl tensor,

$$k_{[e}C_{a]bcd}k^b k^d = 0. \tag{21}$$

Hence the null-homogeneous space–times are algebraically special. A simple direct proof of this fact is based on the relation

$$R_{abcd}k^a k^c = -p_c p^c k_b k_d. \tag{22}$$

The following theorem summarises the results obtained.

*Theorem.* If the energy–momentum tensor of a null-homogeneous space–time obeys the energy conditions (8) and (9), then the normal null congruence $k^a$ must be

(i) non-expanding and shear-free,
(ii) an eigendirection of the Ricci tensor,
(iii) a repeated principal null direction of the Weyl tensor.

The existence of a non-expanding and shear-free normal null congruence implies that the space–time metric can be cast into the normal form (Kundt 1961)

$$\mathrm{d}s^2 = 2P^{-2}|\mathrm{d}\zeta + B\,\mathrm{d}u|^2 - 2\mathrm{d}u\,\mathrm{d}v - 2H\,\mathrm{d}u^2,$$
$$P_{,v} = 0, \qquad B_{,vv} = 0, \tag{23}$$

where $P$ and $H$ are real functions and $B$ is complex. Obviously, there are two-dimensional space-like surfaces $V_2$ ($u$ = constant, $v$ = constant), the wave surfaces. We choose the null tetrad such that $t^a$ and $\bar{t}^a$ are surface-forming vector fields in $V_2$,

$$t^b\bar{t}^a_{,b} = \bar{t}^b t^a_{,b} = At^a - \bar{A}\bar{t}^a. \tag{24}$$

The Gaussian curvature

$$K = -R_{abcd}t^a\bar{t}^b t^c\bar{t}^d = 2P^2(\ln P)_{,\zeta\bar{\zeta}} \tag{25}$$

of $V_2$ is an invariant of the four-dimensional geometry and must have the same value at all points of a hypersurface $N_3$:

$$k_{,a}\xi^a_A = 0 \Rightarrow K = K(u). \tag{26}$$

Not all of Petrov's metrics belong to the class (23), so that the energy conditions rule out some of the gravitational fields admitting a $G_3$ on $N_3$.

Up to now we did not need any specified form of the energy–momentum tensor; only the energy conditions have been used. Perfect fluid solutions are excluded by (13).

For vacuum, Einstein–Maxwell and radiation fields ($T_{ab} = \mu k_a k_b$), it can be shown, with the aid of the Newman–Penrose equations, that the spin coefficient $\tau$ must vanish,

$$\tau \equiv k_{a;b} t^a l^b = -p_a t^a = 0 \qquad (27)$$

$$\Rightarrow p_{[a} k_{b]} = 0 \Rightarrow k_{a;b} = f(u) k_a k_b.$$

Hence there is a covariantly constant null vector field parallel to $k^a$, and the only solutions of the field equations are plane waves.

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## References

Crade R F and Hall G S 1979 *Preprint* University of Aberdeen
Goldberg J N and Sachs R K 1962 *Acta Phys. Pol.* **22** Suppl. 13
Hawking S W and Ellis G F R 1973 *The large scale structure of space–time* (Cambridge: University Press)
Kundt W 1961 *Z. Phys.* **163** 77
Lauten W T and Ray J R 1975 *Lett. Nuovo Cim.* **14** 63
—— 1977 *J. Math. Phys.* **18** 885
MacCallum M A H 1979 *Preprint* Queen Mary College, London
Petrov A Z 1966 *New methods in the theory of General Relativity* (in Russian, Moscow: Nauka) (English edn: Einstein spaces, Pergamon Press 1969)
Ray J R 1976 *Acta Phys. Pol.* **B7** 751